DERIVATION OF AN APPROXIMATE THEORY OF SHELLS BY MEANS OF ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF THE THEORY OF ELASTICITY

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In [1], a method is described for the asymptotic integration of the differential equations of the theory of elasticity which enables us to derive to any required degree of accuracy an approximate theory for the bending of plates. Analogous concepts have been employed (without emphasizing the asymptotic aspect of the approach) in developing various approximate theories of plates in bending and tension, as well as approximate theories of shells [2-6]. The fundamental idea on which such a method is based, the separate and independent consideration of the elastic edge effects, was evolved a little earlier by Friedrichs [7-8], but was applied only to derive the boundary conditions on the free edge of a shell.

In the present study the method [1] is employed to derive a general theory of shells. It will be found that there is a close relationship between the asymptotic method of deriving an approximate theory of shells and the method of asymptotic integration of the differential equations of the theory of shells described in [9].

1. We refer space to an arbitrary curvilinear system of coordinates x^1, x^2, x^3 and denote the radius-vector of an arbitrary point by $\mathbf{R} = \mathbf{R}(x^1, x^2, x^3)$. Then the principal vector $\mathbf{R}_i$ and the metric tensor g_{ij} are defined by the formulas

$$\mathbf{R}_i = \partial \mathbf{R} / \partial x^i, \qquad g_{ij} = \mathbf{R}_i \cdot \mathbf{R}_j$$

Here and in what follows the Latin indices assume the values

1, 2 and 3.

Let $\mathbf{U}$ be the displacement vector, γ_{ij} the strain tensor and σ^{ij} the stress tensor. Then, for an isotropic elastic body

$$\gamma_{ij} = \frac{1}{2}\left(\mathbf{R}_i \cdot \frac{\partial \mathbf{U}}{\partial x^j} + \mathbf{R}_j \cdot \frac{\partial \mathbf{U}}{\partial x^i}\right), \qquad E\gamma_{ij} = (1+\sigma)\, g_{ir}\sigma_j^{\ r} - \sigma g_{ij}\sigma_r^{\ r} \quad (1.1)$$

where E is Young's modulus and σ is Poisson's ratio (the tensor relations used here can be found, for example, in [10]).

In the absence of body forces the equations of equilibrium are

$$\partial \mathbf{T}^i / \partial x^i = 0, \qquad \mathbf{T}^i = \sqrt{g}\,\sigma^{ij}\mathbf{R}_j \qquad (g = |g_{ij}|) \quad (1.2)$$

It is convenient in the theory of shells to make use of curvilinear coordinates in which

$$\mathbf{R} = \mathbf{r}(x^1, x^2) + x^3\mathbf{n}$$

where $\mathbf{r}$ is the radius-vector of the middle surface and $\mathbf{n}$ is a unit vector normal to the middle surface.

The tensors of the first and second quadratic forms of the middle surface will be denoted by $a_{\alpha\beta}$ and $b_{\alpha\beta}$, respectively, and we shall adopt the convention that Greek indices here and in what follows assume the values 1 and 2. The metric tensor g_{ij} can then be expressed in terms of $a_{\alpha\beta}$ and $b_{\alpha\beta}$ as follows:

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x^3 b_{\alpha\beta} + (x^3)^2 b_\alpha^{\ \lambda} b_{\beta\lambda}, \qquad g_{\alpha 3} = 0, \qquad g_{33} = 1 \quad (1.3)$$

In addition, there exist the formulas

$$\sqrt{\frac{g}{a}} = 1 - x^3 b_\lambda^{\ \lambda} + (x^3)^2 K, \qquad a = |a_{\alpha\beta}|, \qquad K = b_1^{\ 1} b_2^{\ 2} - b_2^{\ 1} b_1^{\ 2} \quad (1.4)$$

in which K is the Gaussian curvature of the middle surface.

It is convenient to introduce the non-symmetric stress tensor τ^{ij}

$$\sqrt{g/a}\,\sigma_\beta^{\ i} = (a_{\lambda\beta} - x^3 b_{\lambda\beta})\,\tau^{i\lambda}, \qquad \sqrt{g/a}\,\sigma^{i3} = \tau^{i3} \quad (1.5)$$

The vector $\mathbf{T}^i$, which appears in the equilibrium equation, can then be written as follows:

$$\mathbf{T}^i = \sqrt{a}\,(\tau^{i\lambda}\mathbf{r}_\lambda + \tau^{i3}\mathbf{n}) \quad (1.6)$$

After some manipulation the conditions of symmetry of the tensor σ^{ij} can be reduced to the equalities

$$c_{\lambda\beta}\,(\tau^{\lambda\beta} - x^3 b_\alpha{}^\lambda \tau^{\alpha\beta}) = 0, \qquad \tau^{3\lambda} = \tau^{\lambda 3} - x^3 b_\mu{}^\lambda \tau^{\mu 3} \tag{1.7}$$

where $c_{\lambda\mu}$ is a skew-symmetric discriminant tensor with components

$$c_{\alpha\alpha} = 0, \qquad c_{12} = -c_{21} = \sqrt{a}$$

Substituting (1.6) in the equation of equilibrium, we obtain

$$\nabla_\alpha \tau^{\alpha\beta} - b_\alpha{}^\beta \tau^{\alpha 3} + \frac{\partial \tau^{3\beta}}{\partial x^3} = 0, \qquad \nabla_\alpha \tau^{\alpha 3} + b_{\alpha\beta}\tau^{\alpha\beta} + \frac{\partial \tau^{33}}{\partial x^3} = 0 \tag{1.8}$$

Here ∇_α is the symbol of covariant differentiation in the metric on the middle surface. This operation is defined by the following formulas (see, for example, [11]):

$$\nabla_\lambda A^{\alpha\beta} = \frac{\partial A^{\alpha\beta}}{\partial x^\lambda} + \Gamma_{\mu\lambda}^{\alpha} A^{\mu\beta} + \Gamma_{\mu\lambda}^{\beta} A^{\alpha\mu} \qquad \nabla_\lambda A^\alpha = \frac{\partial A^\alpha}{\partial x^\lambda} + \Gamma_{\mu\lambda}^{\alpha} A^\mu$$
$$\nabla_\lambda A_\alpha = \frac{\partial A_\alpha}{\partial x^\lambda} - \Gamma_{\alpha\lambda}^{\mu} A_\mu, \qquad \nabla_\lambda A = \frac{\partial A}{\partial x^\lambda} \tag{1.9}$$

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{a^{\alpha\lambda}}{2}\left(\frac{\partial a_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial a_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial a_{\beta\gamma}}{\partial x^\lambda}\right) \tag{1.10}$$

(the expression for $\nabla_\alpha \tau^{\alpha 3}$ has to be written out in the second of these formulas). We assume that the displacement vector can be written as

$$\mathbf{U} = u^s \mathbf{r}_s - W\mathbf{n}$$

Then, making use of (1.1), we obtain

$$2\gamma_{\alpha\beta} = \nabla_\beta u_\alpha + \nabla_\alpha u_\beta + 2b_{\alpha\beta}W - x^3\,[b_\beta{}^\lambda(\nabla_\alpha u_\lambda + b_{\lambda\alpha}W) + b_\alpha{}^\lambda(\nabla_\beta u_\lambda + b_{\lambda\beta}W)]$$
$$2\gamma_{\alpha 3} = -\nabla_\alpha W + b_\alpha{}^\lambda u_\lambda + \frac{\partial u_\alpha}{\partial x^3} - x^3 b_\alpha{}^\lambda \frac{\partial u_\lambda}{\partial x^3}, \qquad \gamma_{33} = -\frac{\partial W}{\partial x^3} \tag{1.11}$$

Taking into account (1.5), we can re-write the elastic relations (1.1) as follows:

$$-E\sqrt{\frac{g}{a}}\,\frac{\partial W}{\partial x^3} = \tau^{33} - \sigma\tau^{\alpha\rho}\,(a_{\alpha\rho} - x^3 b_{\alpha\rho}) \tag{1.12}$$

$$\frac{E}{2}\sqrt{\frac{g}{a}}\left(-\nabla_\alpha W + b_\alpha{}^\lambda u_\lambda + \frac{\partial u_\alpha}{\partial x^3} - x^3 b_\alpha{}^\lambda \frac{\partial u_\lambda}{\partial x^3}\right) = (1+\sigma)\, g_{\alpha\lambda}\tau^{\lambda 3}$$

$$\frac{E}{2}\sqrt{\frac{g}{a}}\,\{\nabla_\beta u_\alpha + \nabla_\alpha u_\beta + 2b_{\alpha\beta}W - x^3\,[b_\beta{}^\lambda(\nabla_\alpha u_\lambda + b_{\lambda\alpha}W) +$$
$$+ b_\alpha{}^\lambda(\nabla_\beta u_\lambda + b_{\lambda\beta}W)]\} =$$
$$= (1+\sigma)\, g_{\alpha\lambda}\,(a_{\mu\beta} - x^3 b_{\mu\beta})\,\tau^{\lambda\mu} - \sigma g_{\alpha\beta}\,[(a_{\lambda\mu} - x^3 b_{\lambda\mu})\,\tau^{\lambda\mu} + \tau^{33}]$$

If we denote the thickness of the shell by $2h$, then the external and internal surfaces of the shell will be given by $x^3 = \pm h$. The boundary conditions, which we shall assume to be of the form

$$\tau^{33} = \pm \tfrac{1}{2} x, \qquad \tau^{3\alpha} = \pm \tfrac{1}{2} X^\alpha \quad \text{for } x^3 = \pm h \tag{1.13}$$

must be satisfied on these surfaces.

Here and in the sequel we assume that h is constant.

2. The equilibrium equations (1.8), the conditions of symmetry (1.7) and the elastic relations (1.12) constitute a complete system of differential equations for the determination of the displacements and stresses In order to integrate this system we adopt the method employed in [1] for the formulation of the principal iteration process. We change the independent variables according to the formulas

$$x^\alpha = R\xi^\alpha, \qquad x^3 = h\zeta \tag{2.1}$$

(here R is the characteristic radius of curvature of the middle surface) and we assume that the stresses and displacements do not vary too rapidly with respect to the variables ξ^1, ξ^2, ζ, i.e. we assume that the required state of stress varies rapidly only in the direction of the variable x^3.

After the change of variables (2.1) the foregoing equations assume the form

$$h^*\nabla_\alpha'\tau^{\alpha\beta} - h^*Rb_\alpha{}^\beta\tau^{\alpha 3} + \frac{\partial\tau^{3\beta}}{\partial\zeta} = 0, \qquad h^*\nabla_\alpha'\tau^{\alpha 3} + h^*Rb_{\alpha\beta}\tau^{\alpha\beta} + \frac{\partial\tau^{33}}{\partial\zeta} = 0$$

$$c_{\lambda\beta}(\tau^{\lambda\beta} - h^*\zeta Rb_\alpha{}^\lambda\tau^{\alpha\beta}) = 0, \qquad \tau^{3\lambda} = \tau^{\lambda 3} - h^*\zeta Rb_\mu{}^\lambda\tau^{\mu 3}$$

$$-\frac{E}{R}\sqrt{\frac{g}{a}}\frac{\partial W}{\partial\zeta} = h^*[\tau^{33} - \sigma\tau^{\alpha\rho}(a_{\alpha\rho} - h^*\zeta Rb_{\alpha\beta})] \tag{2.2}$$

$$\frac{E}{2R}\sqrt{\frac{g}{a}}\left(-h^*\nabla_\alpha'W + h^*Rb_\alpha{}^\lambda u_\lambda + \frac{\partial u_\alpha}{\partial\zeta} - h^*\zeta Rb_\alpha{}^\lambda\frac{\partial u_\lambda}{\partial\zeta}\right) = h^*(1+\sigma)g_{\alpha\lambda}\tau^{\lambda 3}$$

$$\frac{E}{2R}\sqrt{\frac{g}{a}}\{\nabla_\beta'u_\alpha + \nabla_\alpha'u_\beta + 2Rb_{\alpha\beta}W - \zeta h^*[Rb_\beta{}^\lambda(\nabla_\alpha'u_\lambda + Rb_{\lambda\alpha}W) +$$

$$+ Rb_\alpha{}^\lambda(\nabla_\beta'u_\lambda + Rb_{\lambda\beta}W)]\} =$$

$$= (1+\sigma)g_{\alpha\lambda}(a_{\mu\beta} - h^*\zeta Rb_{\mu\beta})\tau^{\lambda\mu} - \sigma g_{\alpha\beta}[(a_{\lambda\mu} - \zeta h^*Rb_{\lambda\mu})\tau^{\lambda\mu} + \tau^{33}]$$

where $h^* = h/R$ and $\nabla_\alpha' = R\nabla_\alpha$ is the symbol of covariant differentiation with respect to the variables ξ^α.

Note. The change of variables $x^\alpha = R\xi^\alpha$ in (2.1) is equivalent to changing the symbol ∇_α to $\nabla_\alpha' R^{-1}$. For example, from the first of formulas (1.9)

$$R\nabla_\lambda A^{\alpha\beta} = \frac{\partial A^{\alpha\beta}}{\partial \xi^\lambda} + R\Gamma_{\mu\lambda}^{\alpha} A^{\mu\beta} + R\Gamma_{\mu\lambda}^{\beta} A^{\alpha\mu}$$

and, according to (1.10)

$$R\Gamma_{\beta\gamma}^{\alpha} = \frac{a^{\alpha\lambda}}{2}\left(\frac{\partial a_{\beta\lambda}}{\partial \xi^\gamma} + \frac{\partial a_{\gamma\lambda}}{\partial \xi^\beta} - \frac{\partial a_{\beta\gamma}}{\partial \xi^\lambda}\right)$$

The system (2.2) contains a small parameter h^*. This parameter occurs explicitly in the equations and is contained in the quantities g and $g_{\alpha\beta}$, which, according to (1.3), (1.4) and (2.1), are defined by the formulas

$$\sqrt{\frac{g}{a}} = 1 - h^*\zeta R b_\lambda{}^\lambda + h^{*2}\zeta^2 R^2 K, \qquad g_{\alpha\beta} = a_{\alpha\beta} - 2h^*\zeta R b_{\alpha\beta} + h^{*2}\zeta^2 R^2 b_\alpha{}^\lambda b_{\beta\lambda} \tag{2.3}$$

3. We shall try to find a solution of equations (2.2) in the form

$$\begin{gathered}\tau^{\alpha\beta} = h^{*-r}\Sigma h^{*s}\tau_{(s)}{}^{\alpha\beta}, \qquad \tau^{\alpha 3} = h^{*-r}\Sigma h^{*s}\tau_{(s)}{}^{\alpha 3}, \qquad \tau^{33} = h^{*-r}\Sigma h^{*s}\tau_{(s)}{}^{33} \\ u_\alpha = h^{*-r}\Sigma h^{*s}u_\alpha{}^{(s)}, \qquad W = h^{*-r}\Sigma h^{*s}W^{(s)}\end{gathered} \tag{3.1}$$

Here r is a number (which is different for different quantities) which will be selected later; quantities qualified by the index s (in brackets, since it has no tensorial connotation) either are totally independent of h or they contain a factor h^p which is common to all these quantities; the summation is carried out over all values of s, starting from zero.

We select r in the following manner:

$$\tau^{\alpha\beta} \to r = \varkappa + 1, \qquad (\tau^{\alpha 3}, \tau^{33}) \to r = \varkappa, \qquad (u_\alpha, W) \to r = \varkappa + 1 \tag{3.2}$$

(here κ is for the present unspecified). We then substitute the expansions of (3.1) into (2.2) and make the stipulation that in each equation of (2.2) taken separately, the coefficients of all powers of h^* vanish, starting from the lowest. In this way, we obtain a sequence of systems of equations for determining the coefficients of the expansions of (3.1). The first of these systems is of the form

$$\begin{gathered}\nabla_\alpha{}'\tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \qquad R b_{\alpha\beta}\tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0 \\ c_{\lambda\beta}\tau_{(0)}{}^{\lambda\beta} = 0, \qquad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3} \\ \frac{\partial W^{(0)}}{\partial \zeta} = 0, \qquad \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} = 0\end{gathered} \tag{3.3}$$

$$\frac{E}{2R}(\nabla_\beta' u_\alpha^{(0)} + \nabla_\alpha' u_\beta^{(0)} + 2Rb_{\alpha\beta}W^{(0)}) = P_{\alpha\beta\lambda\mu}\tau_{(0)}^{\lambda\mu}$$

where

$$P_{\alpha\beta\lambda\mu} = (1+\sigma)\, a_{\alpha\lambda}a_{\beta\mu} - \sigma a_{\alpha\beta}a_{\lambda\mu} \tag{3.4}$$

Equations (3.3) may be easily integrated with respect to ζ. Having performed this operation, setting $\kappa = 0$, taking into account the conditions on the surfaces (1.13) and accepting that the quantities x and X^α can be expressed in the form

$$X^\alpha = \Sigma h^{*s} X_{(s)}^{\alpha}, \qquad x = \Sigma h^{*s} x_{(s)} \tag{3.5}$$

we obtain

(3.6)

$$W^{(0)} = w^{(0)}(\xi^1, \xi^2,) \qquad u_\alpha^{(0)} = v_\alpha^{(0)}(\xi^1, \xi^2)$$

$$\frac{E}{2R}(\nabla_\beta' v_\alpha^{(0)} + \nabla_\alpha' v_\beta^{(0)} + 2Rb_{\alpha\beta}w^{(0)}) = P_{\alpha\beta\lambda\mu}\tau_{(0)}^{\lambda\mu}$$

$${}_{\lambda\beta}\tau_{(0)}^{\alpha\beta} = 0, \quad \tau_{(0)}^{3\lambda} = \tau_{(0)}^{\lambda 3}, \quad \nabla_\alpha'\tau_{(0)}^{\alpha\beta} = -\frac{1}{2}X_{(0)}^{\beta}, \quad Rb_{\alpha\beta}\tau_{(0)}^{\alpha\beta} = -\frac{1}{2}x_{(0)}$$

$$\tau_{(0)}^{3\beta} = \frac{\zeta}{2}X_{(0)}^{\beta}, \qquad \tau_{(0)}^{33} = \frac{\zeta}{2}x_{(0)}$$

The equalities (3.6) constitute a complete system of differential equations (with independent variables ξ^1, ξ^2) in the unknowns $\tau_{(0)}^{\alpha\beta}$, $\tau_{(0)}^{3\lambda}$, $\tau_{(0)}^{\lambda 3}$, $w^{(0)}$, $v_\alpha^{(0)}$, which are independent of ζ. Here the stresses $\tau_{(0)}^{\alpha\beta}$ are constant throughout the thickness of the shell, and, as will be shown in Section 11, the corresponding state of stress is closely related to the membrane state of stress in the classical theory of shells.

4. For the homogeneous equations (2.2), when

$$X_{(s)}^{\alpha} = x_{(s)} = 0 \tag{4.1}$$

there exists one further form of the expansions of (3.1). It is obtained by selecting the following combination of values of r

$$\tau^{\alpha\beta} \to r = \varkappa + 1, \qquad (\tau^{\alpha 3}, \tau^{33}) \to r = \varkappa, \qquad (u_\alpha, W) \to r = \varkappa + 2 \tag{4.2}$$

We substitute (3.1) and (4.2) into equations (2.2) and impose the requirements that in the first five of these the coefficients of the lowest power of h^* vanish, and in the sixth and seventh of equations (2.2) the coefficients of the lowest and next lowest powers of h^* vanish. Taking into account (2.3), we obtain

$$\nabla_\alpha'\tau_{(0)}^{\alpha\beta} + \frac{\partial\tau_{(0)}^{3\beta}}{\partial\zeta} = 0, \quad Rb_{\alpha\beta}\tau_{(0)}^{\alpha\beta} + \frac{\partial\tau_{(0)}^{33}}{\partial\zeta} = 0, \quad c_{\lambda\beta}\tau_{(0)}^{\lambda\beta} = 0, \quad \tau_{(0)}^{3\lambda} = \tau_{(0)}^{\lambda 3}$$

$$\frac{\partial W^{(0)}}{\partial \zeta} = 0, \qquad \frac{\partial u_\alpha^{(0)}}{\partial \zeta} = 0, \qquad \frac{\partial u_\alpha^{(1)}}{\partial \zeta} - \nabla_\alpha' W^{(0)} + R b_\alpha^{\ \lambda} u_\lambda^{(0)} = 0$$

$$\frac{1}{2R}\left(\nabla_\beta' u_\alpha^{(0)} + \nabla_\alpha' u_\beta^{(0)} + 2R b_{\alpha\beta} W^{(0)}\right) = 0$$

$$\frac{E}{2R}\left[\nabla_\beta' u_\alpha^{(1)} + \nabla_\alpha' u_\beta^{(1)} - R b_\beta^{\ \lambda} \zeta \left(\nabla_\alpha' u_\lambda^{(0)} + R b_{\lambda\alpha} W^{(0)}\right) - \right.$$

$$\left. - R b_\alpha^{\ \lambda} \zeta \left(\nabla_\beta' u_\lambda^{(0)} + R b_{\lambda\beta} W^{(0)}\right)\right] = P_{\alpha\beta\lambda\mu} \tau_{(0)}^{\lambda\mu} \tag{4.3}$$

Carrying out the integration with respect to ζ in these equations, and taking into account the homogeneous conditions on the surfaces (1.13), we find that

$$W^{(0)} = w^{(0)}(\xi^1, \xi^2), \qquad u_\alpha^{(0)} = v_\alpha^{(0)}(x^1, x^2), \qquad u_\alpha^{(1)} = \zeta\left(\nabla_\alpha' w^{(0)} - R b_\alpha^{\ \lambda} v_\lambda^{(0)}\right)$$

$$\nabla_\beta' v_\alpha^{(0)} + \nabla_\alpha' v_\beta^{(0)} + 2R b_{\alpha\beta} w^{(0)} = 0$$

$$\frac{E\zeta}{2R}\left[\nabla_\beta'\left(\nabla_\alpha' w^{(0)} - R b_\alpha^{\ \lambda} v_\lambda^{(0)}\right) + \nabla_\alpha'\left(\nabla_\beta' w^{(0)} - R b_\beta^{\ \lambda} v_\lambda^{(0)}\right) - \right.$$

$$\left. - R b_\beta^{\ \lambda} \zeta \left(\nabla_\alpha' v_\lambda^{(0)} + R b_{\lambda\alpha} W^{(0)}\right) - R b_\alpha^{\ \lambda} \zeta \left(\nabla_\beta' v_\lambda^{(0)} + R b_{\lambda\beta} W^{(0)}\right)\right] = P_{\alpha\beta\lambda\mu} \tau_{(0}^{\ \lambda\mu}$$

$$\tau_{(0)}^{3\beta} = \frac{1}{2}(1 - \zeta^2)\nabla_\alpha' \frac{\tau_{(0)}^{\alpha\beta}}{\zeta}, \qquad \tau_{(0)}^{33} = \frac{1}{2}(1 - \zeta^2) R b_{\alpha\beta} \frac{\tau_{(0)}^{\alpha\beta}}{\zeta} \tag{4.4}$$

In the fifth equality of (4.4) the tensor $\tau_{(0)}^{\alpha\beta}$ is given as a homogeneous function of ζ. It will be shown in Section 11 that formulas (4.4) define a state of stress closely related to the purely bending state of stress in the classical theory of shells.

5. Let us consider now states of stress and strain which vary rapidly not only with respect to the variable x^3, but also with respect to the variables x^1 and x^2. For this purpose, instead of (2.1), we make the change of independent variables

$$x^\alpha = \frac{R}{k_{(\alpha)}} \xi^\alpha, \qquad x^3 = h\zeta \tag{5.1}$$

and assume that with respect to (ξ^1, ξ^2, ζ) the variation in the required states of stress and strain is not too large.

In formulas (5.1) $k_{(\alpha)}$ is a large (compared to unity) non-dimensional number with increase in which the variation of the states of stress and strain increases. It is convenient, as in [9], to express $k_{(\alpha)}$ in terms of h^* by means of the formula

$$k_{(\alpha)} = (h^*)^{-t_\alpha}$$

in which the number t_α, as defined in [9], is the index of variation, in the direction of the x^α line.

From now on we shall always assume that t_α is a rational number equal to p_α/q_α, where p_α and q_α are positive whole numbers.

Making the change of variables (5.1) in equalities (1.9), we obtain

$$R\nabla_\lambda A^{\alpha\beta} = k_{(\lambda)}\nabla_\lambda{}^* A^{\alpha\beta}, \qquad R\nabla_\lambda A_\alpha = k_{(\lambda)}\nabla_\lambda{}^* A_\alpha, \ldots$$

where

$$\nabla_\lambda{}^* A^{\alpha\beta} = \partial_\lambda A^{\alpha\beta} + \frac{R}{k_{(\lambda)}}(\Gamma_{\mu\lambda}^{\alpha} A^{\mu\beta} + \Gamma_{\mu\lambda}^{\beta} A^{\alpha\mu}) \tag{5.2}$$

$$\nabla_\lambda{}^* A_\alpha = \partial_\lambda A_\alpha - \frac{R}{k_{(\lambda)}}\Gamma_{\alpha\lambda}^{\mu} A_\mu, \ldots$$

and ∂_λ is the symbol of ordinary (not covariant) differentiation with respect to ξ^λ.

We can now express the equilibrium equations (1.8), the conditions of symmetry (1.7) and the elastic relations (1.12) in the form

$$h^* k_{(\alpha)}\nabla_\alpha{}^*\tau^{\alpha\beta} - h^* R b_\alpha{}^\beta \tau^{\alpha 3} + \frac{\partial \tau^{3\beta}}{\partial\zeta} = 0, \qquad h^* k_{(\alpha)}\nabla_\alpha{}^*\tau^{\alpha 3} + h^* R b_{\alpha\beta}\tau^{\alpha\beta} + \frac{\partial\tau^{33}}{\partial\zeta} = 0$$

$$c_{\lambda\beta}(\tau^{\lambda\beta} - h^* R\zeta b_\alpha{}^\lambda \tau^{\alpha\beta}) = 0, \qquad \tau^{3\lambda} = \tau^{\lambda 3} - h^* R\zeta b_\mu{}^\lambda \tau^{\mu 3}$$

$$-\frac{E}{R}\sqrt{\frac{g}{a}}\frac{\partial W}{\partial\zeta} = h^*[\tau^{33} - \sigma\tau^{\alpha\beta}(a_{\alpha\beta} - h^* R\zeta b_{\alpha\beta})] \tag{5.3}$$

$$\frac{E}{2R}\sqrt{\frac{g}{a}}\left(-h^* k_{(\alpha)}\nabla_\alpha{}^* W + h^* R b_\alpha{}^\lambda u_\lambda + \frac{\partial u_\alpha}{\partial\zeta} - h^* R\zeta b_\alpha{}^\lambda \frac{\partial u_\lambda}{\partial\zeta}\right) = h^*(1+\sigma) g_{\lambda\alpha}\tau^{\lambda 3}$$

$$\frac{E}{2R}\sqrt{\frac{g}{a}}\{k_{(\beta)}\nabla_\beta{}^* u_\alpha + k_{(\alpha)}\nabla_\alpha{}^* u_\beta + 2Rb_{\alpha\beta}W -$$

$$- h^*\zeta[Rb_\beta{}^\lambda(k_{(\alpha)}\nabla_\alpha{}^* u_\lambda + Rb_{\lambda\alpha}W) + Rb_\alpha{}^\lambda(k_{(\beta)}\nabla_\beta{}^* u_\lambda + Rb_{\lambda\beta}W)]\} =$$

$$= (1+\sigma) g_{\alpha\lambda}(a_{\mu\beta} - h^* R\zeta b_{\mu\beta})\tau^{\lambda\mu} - \sigma g_{\alpha\beta}[(a_{\lambda\mu} - h^* R\zeta b_{\lambda\mu})\tau^{\lambda\mu} + \tau^{33}]$$

6. We begin the investigation of the states of stress with a non-zero index of variation with the case when the variation is the same in the directions of both coordinate lines. Let

$$k_{(1)} = k_{(2)} = k = (h^*)^{-\frac{p}{q}}, \qquad t_{(1)} = t_{(2)} = t = \frac{p}{q}$$

We introduce the notation

$$\eta = (h^*)^{-\frac{1}{q}} \quad \text{or} \quad h^* = \eta^{-q}, \qquad k = \eta^p \tag{6.1}$$

and try to find a solution to the system (5.3) in the form*

$$\tau^{\alpha\beta} = \eta^r \Sigma \eta^{-s} \tau_{(s)}{}^{\alpha\beta}, \qquad \tau^{\alpha 3} = \eta^r \Sigma \eta^{-s} \tau_{(s)}{}^{\alpha 3}, \qquad \tau^{33} = \eta^r \Sigma \eta^{-s} \tau_{(s)}{}^{33}$$
$$u_\alpha = \eta^r \Sigma \eta^{-s} u_\alpha{}^{(s)}, \qquad W = \eta^r \Sigma \eta^{-s} W^{(s)} \tag{6.2}$$

Here r is a number which is different for different unknowns. It must be selected in such a way that after substituting (6.2) into equations (5.3) and in each equation equating to zero the coefficients of all powers of η starting with the highest, we obtain a non-contradictory sequence of systems of equations for the determination of the coefficients of the expansions of (6.2). Such values of r will be called non-contradictory values.

In seeking non-contradictory values of r we must consider separately the cases when

$$t < 1/2, \qquad t = 1/2, \qquad t > 1/2$$

(these cases also arose [9] in the process of asymptotic integration of the differential equations of the theory of thin shells).

We start with the case when $t < 1/2$, i.e. we assume that

$$2p < q \tag{6.3}$$

Then one of the variants of non-contradictory values of r will be

$$\tau^{\alpha\beta} \to r = \varkappa + q, \qquad \tau^{\alpha 3} \to r = \varkappa + p, \qquad \tau^{33} \to r = \varkappa \tag{6.4}$$
$$u_\alpha \to r = \varkappa - p, \qquad W \to r = \varkappa + q$$

In the manner described above this reduces to a sequence of systems of equations in which the principal system is of the form

$$\partial_\alpha \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \quad R b_{\alpha\beta} \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0, \quad c_{\lambda\beta} \tau_{(0)}{}^{\lambda\beta} = 0, \quad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3} \tag{6.5}$$
$$\frac{\partial W^{(0)}}{\partial \zeta} = 0, \quad \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} = 0, \quad \frac{E}{2R} \{\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)} + 2R b_{\alpha\beta} W^{(0)}\} = P_{\alpha\beta\lambda\mu} \tau_{(0)}{}^{\lambda\mu}$$

In deriving these equations we must substitute for h^* and k in (5.3) according to formula (6.1) and then make use of the expansions of (6.2) and (6.4) and the inequality (6.3). In equalities (6.5), as in the subsequent relations of the first approximation, the symbols of covariant

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* The expansions of (6.2) are generalizations of the expansions of (3.1). The latter can be obtained from the former by putting $p = 0$, $q = 1$.

differentiation ∇_α^* must be replaced by the symbols of simple differentiation ∂_α on the basis of formulas (5.2).

In this case the second variant of non-contradictory values of r wil be

$$\begin{gathered}\tau^{\alpha\beta} \to r = \varkappa + q, \qquad \tau^{\alpha 3} \to r = \varkappa + p, \qquad \tau^{33} \to r = \varkappa \\ u_\alpha \to r = \varkappa + 2q - 3p, \qquad W \to r = \varkappa + 2q - 2p\end{gathered} \tag{6.6}$$

This yields a system for which the principal equations are of the form

$$\begin{gathered}\partial_\alpha \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \qquad R b_{\alpha\beta} \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0, \quad c_{\lambda\beta} \tau_{(0)}{}^{\lambda\beta} = 0, \quad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3} \\ \frac{\partial W^{(0)}}{\sigma \zeta} = 0, \qquad \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} = 0, \qquad \frac{E}{2R} \{\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)} + 2 R b_{\alpha\beta} W^{(0)}\} = 0 \\ \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} - \partial_\alpha W^{(0)} = 0, \qquad \frac{E\zeta}{R} \partial_\alpha \partial_\beta w^{(0)} = P_{\alpha\beta\lambda\mu} \tau_{(0)}{}^{\lambda\mu}\end{gathered} \tag{6.7}$$

7. In the case when

$$2p = q \qquad (t = 1/2) \tag{7.1}$$

the non-contradictory combination of values of r may be written as

$$\begin{gathered}\tau^{\alpha\beta} \to r = \varkappa + 2p, \qquad \tau^{\alpha 3} \to r = \varkappa + p, \qquad \tau^{33} \to r = \varkappa \\ u_\alpha \to r = \varkappa + p, \qquad W \to r = \varkappa + 2p\end{gathered} \tag{7.2}$$

This combination corresponds to a sequence of systems of equations in which the principal system is

$$\begin{gathered}\partial_\alpha \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \qquad \partial_\alpha \tau_{(0)}{}^{\alpha 3} + R b_{\alpha\beta} \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0 \\ c_{\lambda\beta} \tau_{(0)}{}^{\lambda\beta} = 0, \qquad \tau_{(0)}{}^{3\beta} = \tau_{(0)}{}^{\beta 3} \\ \frac{\partial W^{(0)}}{\partial \zeta} = 0, \qquad \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} - \partial_\alpha W^{(0)} = 0 \\ \frac{E}{2R} \{\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)} + 2 R b_{\alpha\beta} W^{(0)}\} = P_{\alpha\beta\lambda\mu} \tau_{(0)}{}^{\lambda\mu}\end{gathered} \tag{7.3}$$

The integration of these equations with respect to ζ can be carried out without difficulty. Setting $\kappa = 0$, assuming that in the conditions on the surface (1.13) the quantities x and X^α can be expressed in the form

$$X^\alpha = \Sigma \eta^{-s} X_{(s)}{}^\alpha, \qquad x = \Sigma \eta^{-s} x_{(s)} \tag{7.4}$$

we obtain

$$W^{(0)} = w^{(0)}(\xi^1, \xi^2), \quad u_\alpha^{(0)} = v_\alpha^{(0)}(\xi^1, \xi^2) + \zeta\partial_\alpha w^{(0)} \tag{7.5}$$

$$\tau_{(0)}{}^{\alpha\beta} = \tau_{0(0)}^{\alpha\beta}(\xi^1, \xi^2) + \zeta\tau_{1(0)}^{\alpha\beta}(\xi^1, \xi^2)$$

$$\frac{1}{2R}\{\partial_\beta v_\alpha{}^{(0)} + \partial_\alpha v_\beta{}^{(0)} + 2Rb_{\alpha\beta}w^{(0)}\} = P_{\alpha\beta\lambda\mu}\tau_{0(0)}^{\lambda\mu}, \qquad \frac{E}{R}\partial_\alpha\partial_\beta w^{(0)} = P_{\alpha\beta\lambda\mu}\tau_{1(0)}{}^{\lambda\mu}$$

$$\partial_\alpha\tau_{(0)}{}^{\alpha\beta} = -\frac{1}{2}X_{(0)}{}^\beta, \quad Rb_{\alpha\beta}\tau_{0(0)}{}^{\alpha\beta} + \frac{1}{3}\partial_\alpha\partial_\beta\tau_{1(0)}^{\alpha\beta} = -\frac{1}{2}x_{(0)}$$

$$\tau_{(0)}{}^{3\beta} = -\zeta\partial_\alpha\tau_{0(0)}^{\alpha\beta} + \frac{1}{2}(1-\zeta^2)\,\partial_\alpha\tau_{1(0)}^{\alpha\beta}$$

$$\tau_{(0)}{}^{33} = \frac{\zeta(1-\zeta^2)}{2}Rb_{\alpha\beta}\tau_{0(0)}^{\alpha\beta} + \frac{1-\zeta^2}{2}Rb_{\alpha\beta}\tau_{1(0)}^{\alpha\beta} + \frac{1}{4}(1-\zeta^2)\,\partial_\lambda X_{(0)}{}^\lambda + \frac{3\zeta}{4}\left(1-\frac{\zeta^2}{3}\right)x_0$$

In these formulas the dependence on ζ is always expressed explicitly. In particular we see that the stresses $\tau_{(0)}{}^{\alpha\beta}$ vary over the thickness of the shell according to an arbitrary linear law. This means that formulas (7.5) define a compound state of stress in which the stresses due to forces and moments are commensurable.

8. In the case when the inequality

$$2p > q \qquad (t > 1/2) \tag{8.1}$$

holds, two non-contradictory combinations of values of r can be found.

The first of these may be written as

$$\begin{gathered}\tau^{\alpha\beta} \to r = \varkappa + q, \quad \tau^{\alpha 3} \to r = \varkappa + p, \quad \tau^{33} \to r = \varkappa + 2p - q \\ u_\alpha \to r = \varkappa + q - p, \quad W \to r = \varkappa + 2q - 2p\end{gathered} \tag{8.2}$$

and yields a sequence of systems of equations, the principal one of which is

$$\partial_\alpha\tau_{(0)}{}^{\alpha\beta} + \frac{\partial\tau_{(0)}{}^{3\beta}}{\partial\zeta} = 0, \quad \partial_\alpha\tau_{(0)}{}^{\alpha 3} + \frac{\partial\tau_{(0)}{}^{33}}{\partial\zeta} = 0, \quad c_{\lambda\beta}\tau_{(0)}{}^{\lambda\beta} = 0, \quad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3} \tag{8.3}$$

$$\frac{\partial W^{(0)}}{\partial\zeta} = 0, \quad \frac{\partial u_\alpha{}^{(0)}}{\partial\zeta} - \partial_\alpha W^{(0)} = 0, \quad \frac{E}{2R}(\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)}) = P_{\alpha\beta\lambda\mu}\tau_{(0)}{}^{\lambda\mu}$$

The second non-contradictory combination of values of r is

$$\begin{gathered}\tau^{\alpha\beta} \to r = \varkappa + q, \quad \tau^{\alpha 3} \to r = \varkappa + p, \quad \tau^{33} \to r = \varkappa + 2p - q \\ u_\alpha \to r = \varkappa + q - p, \quad W \to \varkappa < r < \varkappa + 2q - 2p\end{gathered} \tag{8.4}$$

This results in a sequence of systems of equations in which the principal system is

$$\partial_\alpha \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \qquad \partial_\alpha \tau_{(0)}{}^{\alpha 3} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0, \qquad c_{\lambda\beta} \tau_{(0)}{}^{\lambda 3} = 0, \qquad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3}$$

$$\frac{\partial W^{(0)}}{\partial \zeta} = 0, \qquad \frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} = 0, \; \frac{E}{2R} (\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)}) = P_{\alpha\beta\lambda\beta} \tau_{(0)}{}^{\lambda\mu} \tag{8.5}$$

The range of applicability of equations (8.3) and (8.5) is limited, besides (8.1), by the inequality

$$p < q \qquad (t < 1) \tag{8.6}$$

If $p = q(t = 1)$, then the non-contradictory combination (8.2) of values of r is valid, but corresponds to a sequence of systems of equations the principal system of which is

$$\partial_\alpha \tau_{(0)}{}^{\alpha\beta} + \frac{\partial \tau_{(0)}{}^{3\beta}}{\partial \zeta} = 0, \qquad \partial_\alpha \tau_{(0)}{}^{\alpha 3} + \frac{\partial \tau_{(0)}{}^{33}}{\partial \zeta} = 0, \qquad c_{\lambda\beta} \tau_{(0)}{}^{\lambda\beta} = 0, \qquad \tau_{(0)}{}^{3\lambda} = \tau_{(0)}{}^{\lambda 3}$$

$$E \frac{\partial W^{(0)}}{\partial \zeta} = \tau_{(0)}{}^{33} - \sigma a_{\alpha\beta} \tau_{(0)}{}^{\alpha\beta}, \qquad \frac{E}{2R} \left(\frac{\partial u_\alpha{}^{(0)}}{\partial \zeta} - \partial_\alpha W^{(0)} \right) = (1 + \sigma)\, a_{\alpha\lambda} \tau_{(0)}{}^{\lambda 3} \tag{8.7}$$

$$\frac{E}{2R} (\partial_\beta u_\alpha{}^{(0)} + \partial_\alpha u_\beta{}^{(0)}) = P_{\alpha\beta\lambda\mu} \tau_{(0)}{}^{\lambda\mu} - \sigma a_{\alpha\beta} \tau^{33}$$

These equations differ from the general equations of the theory of elasticity only in that in the first place (8.7) omits terms which depend on the curvature tensor, and secondly, the symbols of covariant differentiation in (8.7) are replaced by symbols of simple differentiation. This means that the inequality (8.6) establishes the limit of the range of applicability of the theory of thin shells within the usual context of this theory. The possibility remains only of making simplifications associated with ignoring the effect of the curvature of the middle surface and with replacing covariant differentials by simple differentials.

9. We shall now consider states of stress with different variations in the directions of the x^1- and x^2-lines, and show that in the case when max $(t_{(1)}, t_{(2)}) = 1$ there exist states of stress which are essentially different from those considered so far. We shall confine our attention to the case when the middle surface of the shell is referred to orthogonal coordinates and when the smallest index of variation is zero. We can then take

$$k_{(1)} = h^{*-1} = \eta, \qquad k_{(2)} = \eta^\circ$$

(in order to be precise we assume that the greater variation occurs along the x^1-line). In this case, for a solution of the type (6.2), there exist two non-contradictory combinations of values of r. The first of these is

$$(\tau^{11}, \tau^{22}, \tau^{33}, \tau^{13}, \tau^{31}) \to r = q - 1, \qquad (\tau^{12}, \tau^{21}, \tau^{23}, \tau^{32}) \to r = q$$
$$(u_1, W) \to r = q - 2, \qquad u_2 \to r = q - 1 \tag{9.1}$$

The principal equations of the corresponding sequence of systems of equations are

(9.2)

$$\partial_1\tau_{(0)}{}^{11} + \partial_2\tau_{(0)}{}^{21} - Rb_2{}^1\tau_{(0)}{}^{23} + \frac{\partial\tau_{(0)}{}^{31}}{\partial\zeta} = 0, \quad \partial_1\tau_{(0)}{}^{12} + \frac{\partial\tau_{(0)}{}^{32}}{\partial\zeta} = 0$$

$$\partial_1\tau_{(0)}{}^{13} + \partial_2\tau_{(0)}{}^{23} + R\,(b_{12}\tau_{(0)}{}^{12} + b_{21}\tau_{(0)}{}^{21}) + \frac{\partial\tau_{(0)}{}^{33}}{\partial\zeta} = 0$$

$$\tau_{(0)}{}^{12} = \tau_{(0)}{}^{21}, \qquad \tau_{(0)}{}^{31} = \tau_{(0)}{}^{13} - \zeta Rb_2{}^1\tau_{(0)}{}^{23}, \qquad \tau_{(0)}{}^{32} = \tau_{(0)}{}^{23}$$

$$-\frac{E}{R}\frac{\partial W^{(0)}}{\partial\zeta} = \tau_{(0)}{}^{33} - \sigma\,[a_{11}\tau_{(0)}{}^{11} + a_{22}\tau_{(0)}{}^{22} - \zeta Rb_{12}\tau_{(0)}{}^{12} - \zeta Rb_{21}\tau_{(0)}{}^{21}]$$

$$\frac{E}{2R}\left[-\partial_1 W^{(0)} + Rb_1{}^2u_2{}^{(0)} + \frac{\partial u_1{}^{(0)}}{\partial\zeta} - \zeta Rb_1{}^2\frac{\partial u_2{}^{(0)}}{\partial\zeta}\right] =$$
$$= (1 + \sigma)\,a_{11}\,(\tau_{(0)}{}^{13} - 2Rb_{12}\zeta\tau^{23})$$

$$\frac{E}{2R}\frac{\partial u_2{}^{(0)}}{\partial\zeta} = (1 + \sigma)\,a_{22}\tau_{(0)}{}^{23}$$

$$\frac{E}{R}\,[\partial_1u_1{}^{(0)} - \zeta Rb_1{}^2\partial_1u_2{}^{(0)}] = a_{11}a_{11}\tau_{(0)}{}^{11} - 2\zeta Ra_{11}b_{21}\tau_{(0)}{}^{12} -$$
$$- \sigma a_{11}a_{22}\tau_{(0)}{}^{22} - \sigma a_{11}\tau_{(0)}{}^{33}$$

$$\frac{E}{2R}\partial_1u_2{}^{(0)} = (1 + \sigma)\,a_{11}a_{22}\tau_{(0)}{}^{12}$$

$$\frac{E}{R}\,\partial_2u_2{}^{(0)} = a_{22}a_{22}\tau_{(0)}{}^{22} - \sigma a_{22}a_{11}\tau_{(0)}{}^{11} - \sigma a_{22}\tau_{(0)}{}^{33} - (3 + \sigma)\,\zeta Rb_{12}a_{22}\tau_{(0)}{}^{21}$$

The second non-contradictory combination of values of r may be written as

$$(\tau^{11}, \tau^{22}, \tau^{33}, \tau^{13}, \tau^{31}) \to r + q, \quad (\tau^{12}, \tau^{21}, \tau^{23}, \tau^{32}) \to r = q - 1$$
$$(u_1, W) \to r = q - 1, \quad u_2 \to r = q - 2 \tag{9.3}$$

The principal equations of the corresponding sequence of systems of equations in this case is

(9.4)

$$\partial_1\tau_{(0)}{}^{11} + \frac{\partial\tau_{(0)}{}^{31}}{\partial\zeta} = 0, \quad \partial_1\tau_{(0)}{}^{12} + \partial_2\tau_{(0)}{}^{22} - Rb_1{}^2\tau_{(0)}{}^{13} + \frac{\partial\tau_{(0)}{}^{32}}{\partial\zeta} = 0$$

$$\partial_1\tau_{(0)}{}^{13} + \frac{\partial\tau_{(0)}{}^{33}}{\partial\zeta} = 0$$

$$c_{12}\tau_{(0)}{}^{12} + c_{21}\tau_{(0)}{}^{21} - \zeta R\,[c_{12}b_2{}^1\tau_{(0)}{}^{22} + c_{21}b_1{}^2\tau_{(0)}{}^{11}] = 0$$

$$\tau_{(0)}{}^{31} = \tau_{(0)}{}^{13}, \qquad \tau_{(0)}{}^{32} = \tau_{(0)}{}^{23} - \zeta R b_1{}^2 \tau_{(0)}{}^{13}$$

$$-\frac{E}{R}\frac{\partial W^{(0)}}{\partial \zeta} = \tau_{(0)}{}^{33} - \sigma\,[a_{11}\tau_{(0)}{}^{11} + a_{22}\tau_{(0)}{}^{22}]$$

$$\frac{E}{2R}\left(-\,\partial_1 W^{(0)} + \frac{\partial u_1{}^{(0)}}{\partial \zeta}\right) = (1+\sigma)\,a_{11}\tau_{(0)}{}^{13}$$

$$\frac{E}{2R}\left(-\,\partial_2 W^{(0)} + R b_2{}^1 u_1{}^{(0)} + \frac{\partial u_2{}^{(0)}}{\partial \zeta} - \zeta R b_2{}^1\,\frac{\partial u_1{}^{(0)}}{\partial \zeta}\right) =$$
$$= (1+\sigma)\,(-\,2\zeta R b_{12}\tau_{(0)}{}^{13} + a_{22}\tau_{(0)}{}^{23})$$

$$\frac{E}{R}\,\partial_1 u_{1(0)} = a_{11}\,(a_{11}\tau_{(0)}{}^{11} - \sigma a_{22}\tau_{(0)}{}^{22} - \sigma\tau_{(0)}{}^{33})$$

$$\frac{E}{2R}\,(\partial_2 u_1{}^{(0)} + \partial_1 u_2{}^{(0)} + 2b_{12}W^{(0)} - \zeta R b_2{}^1\partial_1 u_1{}^{(0)}) = -\,(1-\sigma)\,\zeta R b_{12}a_{11}\tau_{(0)}{}^{11} +$$
$$+\,(1+\sigma)\,a_{11}a_{22}\tau_{(0)}{}^{12} - 2\zeta\,R b_{12}a_{22}\tau_{(0)}{}^{22} + 2\sigma\zeta R b_{12}\tau_{(0)}{}^{33}$$

$$0 = a_{22}a_{22}\tau_{(0)}{}^{22} - \sigma a_{11}a_{22}\tau_{(0)}{}^{11} - \sigma a_{22}\tau_{(0)}{}^{33}$$

In the system (9.2) the second, fourth, sixth, ninth and eleventh equalities comprise an independent sub-system of equations in the five unknowns $\tau_{(0)}{}^{12}$, $\tau_{(0)}{}^{21}$, $\tau_{(0)}{}^{23}$, $\tau_{(0)}{}^{32}$ and $u_2{}^{(0)}$. In effect they are identical to the equations of the classical problem of the torsion of a rod about the ξ^2-axis. In the system (9.4) the first, third, fifth, seventh, eighth, tenth and twelfth equations comprise a sub-system of equations in the seven unknowns

$$\tau_{(0)}{}^{11},\ \tau_{(0)}{}^{22},\ \tau_{(0)}{}^{33},\ \tau_{(0)}{}^{13},\ \tau_{(0)}{}^{31},\ u_1{}^{(0)},\ W^{(0)}$$

In effect they coincide with the equations of the problem of plane deformation (in the plane of ξ^1, ζ).

Thus the states of stress corresponding to the non-contradictory combinations (9.1) and (9.3) have the same meaning as those obtained by the auxiliary iteration processes [1] in the formulation of the theory of bending of plates. The principal difference between these states of stress and those derived in Sections 3 to 8 is that the derivation of the former reduces to the integration of the differential equations with respect to the variables (ξ^1, ζ), and the derivation of the latter reduces to the integration of the differential equations with respect to the variables (ξ^1, ξ^2).

10. Consider the tensor of tangential forces $T^{\alpha\beta}$, the tensor of moments $M^{\alpha\beta}$ and the vector of shear forces N^α

$$T^{\alpha\beta} = \int_{-h}^{+h} \tau^{\alpha\beta} dx^3, \qquad M^{\alpha\beta} = \int_{-h}^{+h} \tau^{\alpha\beta} x^3 dx^3, \qquad N^{\alpha} = \int_{-h}^{+h} \tau^{\alpha 3} dx^3 \qquad (10.1)$$

We replace x^3 in (10.1) by ζ according to formula (2.1), substitute the expansions of (6.2) for $\tau^{\alpha\beta}$ and $\tau^{\alpha 3}$ and then, taking into account formulas (6.1), we can write (10.1) in the form

$$T^{\alpha\beta} = \eta^{r-q} \Sigma \eta^{-s} T_{(s)}{}^{\alpha\beta}, \qquad M^{\alpha\beta} = \eta^{r-q} \Sigma \eta^{-s} M_{(s)}{}^{\alpha\beta}, \qquad N^{\alpha} = \eta^{r-p} \Sigma \eta^{-s} N_{(s)}{}^{\alpha}$$

where (10.2)

$$T_{(s)}{}^{\alpha\beta} = R \int_{-1}^{+1} \tau_{(s)}{}^{\alpha\beta} d\zeta, \qquad M_{(s)}{}^{\alpha\beta} = R^2 \eta^{-q} \int_{-1}^{+1} \tau_{(s)}{}^{\alpha\beta} \zeta \, d\zeta = R^2 h^* \int_{-1}^{+1} \tau_{(s)}{}^{\alpha\beta} \zeta \, d\zeta$$

$$N_{(s)}{}^{\alpha} = R \eta^{-q+p} \int_{-1}^{+1} \tau_{(s)}{}^{\alpha 3} d\zeta = R h^* k \int_{-1}^{+1} \tau_{(s)}{}^{\alpha 3} d\zeta \qquad (10.3)$$

In the first two formulas of (10.2) we must assign to r the value which it assumes for $\tau^{\alpha\beta}$ and in the third formula, the value which it assumes for $\tau^{\alpha 3}$. Formulas (10.2) and (10.3) hold both for $t > 0$ and for $t = 0$; in the latter case we must take $p = 0$, $q = 1$ and $k = 1$.

We introduce the quantities $v_\alpha^{[s]}$, $w^{[s]}$ (upper indices in square brackets), which are defined by the formulas

$$v_\alpha^{(s)} = \eta^{-\rho q + p} v_\alpha^{[s]} = h^{*\rho} k v_\alpha^{[s]}, \qquad w^{(s)} = \eta^{-\rho q} w^{[s]} = h^{*\rho} w^{[s]} \qquad (10.4)$$

The number ρ which occurs in these formulas assumes different values for different cases.

It will be shown in Section 11 that if $T_{(0)}{}^{\alpha\beta}$, $M_{(0)}{}^{\alpha\beta}$, $N_{(0)}{}^{\alpha}$, $v_\alpha^{[0]}$ and $w^{[0]}$ are identified, respectively, with the tensor of tangential forces, the tensor of moments, the vector of shear forces, the vector of tangential displacements and the normal displacement, then a simple physical interpretation can be ascribed to the principal equations of the various iteration processes derived above which is closely related to the results obtained [9] by means of asymptotic integration of the differential equations of the classical theory of thin elastic shells.

11. Consider equations (3.3), i.e. the principal system of equations of the iteration process described in Section 3. Integrating the first, second, third and seventh of these equations with respect to ζ over the interval $(-1, +1)$, we obtain

$$\nabla_\alpha T_{(0)}{}^{\alpha\beta} + X_{(0)}{}^\beta = 0, \qquad b_{\alpha\beta} T_{(0)}{}^{\alpha\beta} + x_{(0)} = 0, \qquad c_{\alpha\beta} T_{(0)}{}^{\alpha\beta} = 0$$

$$2Eh \, \tfrac{1}{2} \left(\nabla_\beta v_\alpha{}^{[0]} + \nabla_\alpha v_\beta{}^{[0]} + 2 b_{\alpha\beta} w^{[0]}\right) = P_{\alpha\beta\lambda\mu} T_{(0)}{}^{\lambda\mu} \tag{11.1}$$

The procedure for deriving these equalities is as follows. The integrals with respect to ζ are either replaced according to (10.3) or the integration is carried out, and in the substitution of the limits the conditions on the surfaces (1.13) are imposed. The quantities $u_\alpha{}^{(0)}$ and $W^{(0)}$ are expressed in terms of $v_\alpha{}^{(0)}$ and $w^{(0)}$ according to (3.6), and $v_\alpha{}^{(0)}$ and $w^{(0)}$ are expressed in terms of $v_\alpha{}^{[0]}$ and $w^{[0]}$ according to formulas (10.4) with $\rho = 1$, $k = 1$. Finally, $\nabla_\alpha{}'$ corresponding to the variables ξ^α is replaced by $R\nabla_\alpha$ corresponding to the variables x^α.

Similarly, the following equalities are derived from (4.3):

$$\nabla_\beta v_\alpha{}^{[0]} + \nabla_\alpha v_\beta{}^{[0]} + 2R b_{\alpha\beta} w^{[0]} = 0$$

$$\frac{2Eh^3}{3} \frac{1}{2} [\nabla_\beta (\nabla_\alpha w^{[0]} - b_\alpha{}^\lambda v_\lambda{}^{[0]}) + \nabla_\alpha (\nabla_\beta w^{[0]} - b_\beta{}^\lambda v_\lambda{}^{[0]}) - \tag{11.2}$$

$$- b_\beta{}^\lambda (\nabla_\alpha v_\lambda{}^{[0]} + b_{\lambda\alpha} w^{[0]}) - b_\alpha{}^\lambda (\nabla_\beta v_\lambda{}^{[0]} + b_{\lambda\beta} w^{[0]})] = P_{\alpha\beta\lambda\mu} M_{(0)}{}^{\alpha\beta}$$

The first of these is obtained directly from the eighth equality of (4.3) and the second, from the ninth equality of (4.3), which must first be multiplied by ζ and then integrated over the interval $(-1, +1)$. The transformation is carried out as before by making use of formulas (4.4), (10.2) and (10.4); in the latter we take $\rho = 2$, $k = 1$.

The following equalities follow from equations (7.3)

$$\frac{\partial T_{(0)}{}^{\alpha\beta}}{\partial x^\alpha} + k X_{(0)}{}^\beta = 0, \quad b_{\alpha\beta} T_{(0)}{}^{\alpha\beta} + \frac{\partial N_{(0)}{}^\alpha}{\partial x^\alpha} + x = 0, \quad c_{\lambda\beta} T_{(0)}{}^{\lambda\beta} = 0$$

$$\frac{\partial M_{(0)}{}^{\alpha\beta}}{\partial x^\alpha} - N_{(0)}{}^\alpha = 0 \tag{11.3}$$

$$2Eh \frac{1}{2} \left\{ \frac{\partial v_\beta{}^{[0]}}{\partial x^\alpha} + \frac{\partial v_\alpha{}^{[0]}}{\partial x^\beta} + 2 b_{\alpha\beta} w^{[0]} \right\} = P_{\alpha\beta\lambda\mu} T_{(0)}{}^{\lambda\mu}, \qquad \frac{2Eh^3}{3} \frac{\partial^2 w^{[0]}}{\partial x^\alpha \partial x} = P_{\alpha\beta\lambda\mu} M_{(0)}{}^{\alpha\beta}$$

These are derived as follows, The first, second, third and seventh of equalities (7.3) are integrated with respect to ζ over the interval $(-1, +1)$, and in addition, the first and seventh of equalities (7.3) are multiplied by ζ and integrated with respect to ζ over the interval $(-1, +1)$. The equalities so obtained are transformed in the manner described above. In using formulas (10.4) we take $\rho = 1$, bearing in mind that equalities (7.3) hold when $k_{(1)} = k_{(2)} = k = (h^*)^{-1/2}$ and that in (7.3), according to (5.1)

$$\partial_\alpha = \frac{\partial}{\partial \xi^\alpha} = \frac{R}{k}\frac{\partial}{\partial x^\alpha}$$

In order to compare relations (11.1) to (11.3) with the equations of the classical theory of shells, we write the latter as follows:

The equations of equilibrium

$$\nabla_\lambda T^{\lambda\alpha} - b_\lambda{}^\alpha N^\lambda + X^\alpha = 0, \quad b_{\alpha\beta}T^{\alpha\beta} + \nabla_\lambda N^\lambda + x = 0, \quad \nabla_\lambda M^{\alpha\lambda} - N^\alpha = 0$$
$$c_{\alpha\beta}T^{\alpha\beta} - c_\alpha{}^\lambda b_{\lambda\beta}M^{\alpha\beta} = 0 \tag{11.4}$$

The strain-displacement relations

$$\varepsilon_{\alpha\beta} = \nabla_\alpha v_\beta + b_{\alpha\beta}w + c_{\beta\alpha}\delta, \qquad \mu_{\alpha\beta} = \nabla_\beta(\nabla_\alpha w - b_\alpha{}^\lambda v_\lambda) - c_{\lambda\alpha}b_\beta{}^\lambda\delta$$
$$\delta = -\tfrac{1}{2}c^{\alpha\beta}b\nabla_\alpha v_\beta \tag{11.5}$$

The relations between strains, forces and moments

$$\varepsilon_{\alpha\beta} = \frac{1}{2Eh}(P_{\alpha\beta\lambda\mu}T^{\lambda\mu} + Q_{\alpha\beta\lambda\mu}M^{\lambda\mu}), \qquad \mu_{\alpha\beta} = \frac{1}{2Eh}\left(R_{\alpha\beta\lambda\mu}M^{\lambda\mu} + \frac{h^2}{3}S_{\alpha\beta\lambda\mu}T^{\lambda\mu}\right)$$
$$P_{\alpha\beta\lambda\mu} = R_{\alpha\beta\lambda\mu} = a_{\alpha\lambda}a_{\beta\mu} - \sigma c_{\alpha\lambda}c_{\beta\mu} \tag{11.6}$$

These relations are taken from [12] (x in the equations of equilibrium has been replaced by $-x$ in accordance with our present sign convention). There the force and moment tensors are introduced by means of the equalities

$$T^{\alpha\beta} = \int_{-h}^{+h}(\tau^{\alpha\beta} - x^3 b_\lambda{}^\alpha\tau^{\lambda\beta})\sqrt{\frac{g}{a}}\,dx^3, \qquad N^\alpha = \int_{-h}^{+h}\tau^{3\alpha}\sqrt{\frac{g}{a}}\,dx^3$$
$$M^{\alpha\beta} = \int_{-h}^{+h}(\tau^{\alpha\beta} - x^3 b_\lambda{}^\alpha\tau^{\lambda\beta})\sqrt{\frac{g}{a}}\,x^3dx^3,$$

(the minus sign included in error [12] in the formula for $M^{\alpha\beta}$ has been omitted). If we take into account (1.5) we can easily see that these expressions coincide with equalities (10.1).

The first three equations of (11.1) are identical with the equations of equilibrium of the membrane theory, which can be obtained from (11.4) with $M^{\alpha\lambda} = N^\alpha = 0$.

The formula for the tensor of tangential strain $\varepsilon_{\alpha\beta}$ in (11.5) can be reduced to the form

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(\nabla_\beta v_\alpha + \nabla_\alpha v_\beta + 2b_{\alpha\beta}w)$$

and the formula for $P_{\alpha\beta\lambda\mu}$ in (11.6), to the form (3.4). Thus, in (11.6) the tensor $P_{\alpha\beta\lambda\mu}$ has the same meaning as in the other formulas of this paper, and consequently the fourth equality of (11.1) is equivalent to the elastic relations of the membrane theory.

Thus the iteration process derived in Section 3 is equivalent in first approximation to the membrane theory.

It follows from formulas (11.5) that

$$2\mu_{\alpha\beta} - b_\alpha{}^\lambda\varepsilon_{\lambda\beta} - b_\beta{}^\lambda\varepsilon_{\lambda\alpha} = \nabla_\beta\,(\nabla_\alpha w - b^\lambda{}_\alpha v_\lambda) + \nabla_\alpha\,(\nabla_\beta w - b_\beta{}^\lambda v_\lambda) -$$
$$- b_\beta{}^\lambda\,(\nabla_\alpha v_\lambda + b_{\lambda\alpha} w) - b_\alpha{}^\lambda\,(\nabla_\beta v_\lambda + b_{\lambda\beta} w)$$

The left-hand side of this equality with $\varepsilon_{\alpha\beta} = 0$ coincides with $2\mu_{\alpha\beta}$, and the expression on the right-hand side is identical with the expression in square brackets in the second equality of (11.2).

It follows that the iteration process described in Section 4 defines a state of stress which in first approximation is equivalent to a purely bending state of stress.

The first of equalities (11.2) is equivalent to the equality $\varepsilon_{\alpha\beta} = 0$ and constitutes equations of infinitely small bending deformations. To the accuracy of quantities of the order of h^2, the second equality coincides for such a state of stress with the second strain-force-moment relation of (11.6).

The iteration process developed in Section 7 coincides in first approximation with states of stress which have large variations. Indeed, relations (11.3), which are derived from this process, can be obtained from the relations of the classical theory of shells by:

a) discarding the tensor N^α in the first of the equilibrium equations (11.4);

b) discarding tangential displacements in the expression for the strain tensor $\mu_{\alpha\beta}$;

c) accepting the most simple version of the strain-force-moment formulas;

d) substituting simple differentiation for covariant differentiation.

This, in fact, constitutes the familiar hypotheses of the theory of a state of stress with a large variation.

It is likewise not difficult to give a physical interpretation of the principal equations of the iteration processes described in

Section 6 and 8.

Equations (6.5) and (6.7) differ, respectively, from equations (3.3) and (4.3) only in that in the former the symbols of covariant differentiation ∇_α' have been replaced by those for simple differentiation ∂_α, and on the left-hand side of the last equality of (6.7) only those terms remain that contain second derivatives of the normal deflection. The same simplifications result if we use the method of asymptotic integration to find the membrane and purely bending states of stress with non-zero variation and retain in the equations only those terms required for a first approximation. Thus, equations (6.5) and (6.7) correspond to the equations of the first asymptotic approximation of the membrane and purely bending theories of states of stress with non-zero variation.

Similarly, it can be shown that equations (8.3) and (8.5) correspond to the theory of states of stress with large ($t > 1/2$) variation. Equations (8.3) correspond (in the context of [9]) to a state of bending stress similar to that in the bending of a plate. Equations (8.5) correspond to a state of tangential stress similar to a generalized state of plane stress.

12. In Sections 3 to 8 iteration processes are formulated for deriving states of stress which in the zeroth approximation are equivalent to: a membrane state of stress (Sections 3 and 6), a purely bending state of stress (Sections 4 and 6) and a state of stress with a large variation index (Sections 7 and 8). It is not difficult to formulate iteration processes also for the derivation of states of stress corresponding to simple and generalized edge effects. These can be obtained by the integration of equations (5.3) with the aid of the expansions of (6.2) and with the proper choice for the values of r.

The integrals corresponding to all the states of stress enumerated above can also be found directly from the equations of the classical theory of shells by the method of asymptotic integration [9]. Taken together they contain sufficient arbitrary constants to satisfy all four boundary conditions of the classical theory of shells for a wide class of problems (this question has been discussed in [9, 13-15]). For this class of problems of the theory of shells the iteration processes described here play the same part when taken as a set as the basic iteration process in the theory of the bending of plates [1], in the sense that the initial approximation is equivalent to the classical theory. At the same time, in Section 9 iteration processes are formulated which are equivalent to two variants of the auxiliary iteration process [1]. One of them corresponds to the state of stress for edge torsion and the other corresponds to the state of stress for plane edge deformation. By analogy with the theory of plates it is to be expected that by combining the iteration processes of Sections 3 to 8 with those of Section 9 we shall be able to satisfy the boundary conditions of the three-dimensional theory of elasticity to any degree of accuracy for the class of shell-problems considered here. In [1] this question is considered for plates for cases corresponding to a free, a clamped and a simply-supported edge.

13. For all the iteration processes in the present article only the principal systems of equations are given. The derivation of the equations which define the nth terms of the expansions does not present any difficulty (they differ from those given only in their absolute terms), but to include them in full would have been too tedious a task.

It is of interest to make an estimate of the remainders and to formulate the conditions for which the iteration processes described in this study are asymptotic. There are certain mathematical difficulties, but it is to be hoped that these will prove to be not too serious.

The conditions which ensure the asymptotic convergence of the processes studied here define the field of applicability of the results obtained. Some of these conditions are obvious without any mathematical analysis.

One such condition is that the equations which define the initial approximations must have finite solutions for given boundary conditions. This condition is not fulfilled for all cases of practical importance. For instance, the equations of the membrane theory (11.1) do not have finite solutions:

a) for an infinitely long cylindrical shell, for a conical shell having an apex, or in general, for shells which contain a cusp;

b) for a shell which is tangential to a plane along a closed curve (for example a torus).

Such shells, and shells which differ very slightly from them (for example, a very long cylindrical shell), were described in [9] as shells with a singular middle surface. For them the three-dimensional equations of the theory of elasticity must be reduced to two-dimensional equations by other iteration processes.

Equations (11.1) will also not have a solution when $b_{\alpha\beta} = 0$. This means that the question of shallow shells requires a special treatment. This question can evidently be solved by means of the iteration process described in Section 7.

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